Sectional operators, new integrable systems and semidirect Lie algebras

1995 J. Phys. A: Math. Gen. 28 3511

(http://iopscience.iop.org/0305-4470/28/12/022)

J. Phys. A: Math. Gen. 28 (1995) 3511–3523. Printed in the UK

Sectional operators, new integrable systems and semidirect Lie algebras

I Mukhopadhyay and A Roy Chowdhury

High Energy Physics Division, Department of Physics, Jadavpur University, Calcutta-700 032, India

Received 25 August 1994, in final form 3 January 1995

Abstract. Integrable dynamical systems are constructed on semi-direct non-compact Lie algebra $SO(6)\times v_6$, where v_6 is the topological vector space on which the representation of $SO(6)$ is realized. The construction is done with the help of the sectional operators introduced by Trofimov and Fomenko. Our equations generalize those of a rigid body immersed in a fluid and those of Manakov on $SO(6)$. Our equations contain the system of two interacting rigid bodies immersed in a fluid, as a special case.

1. Introduction

Lie algebra is nowadays one of the most important tools for exploring the properties of nonlinear integrable systems [1]. An interesting feature is that it can either be used to solve the nonlinear system or it can be used to construct it [2]. An important landmark within such attempts is the celebrated theorem due to Adler–Kostant–Symes [3]. It has already been observed that whether the Lie algebra is finite- or infinite-dimensional the AKS theorem can always be used to construct the integrable equation. In this respect it may be mentioned that the complete integrability of such nonlinear dynamical systems is always guaranteed due to the existence of Liouville's theorem, which demands the existence of an infinite number of conserved quantities in involution. When the Lie algebra under consideration is either simple or semi-simple the method has been widely discussed and applied to several situations. However, some new complications arise if the Lie algebra is a semi-direct one, i.e. if we consider the semi-direct product of a Lie algebra and a topological vector space V, which serves as the representation space of the algebra. If one considers such a Lie algebra as the starting point then some additional complications do arise [4]. Such a situation was discussed explicitly by Trofimov and Fomenko [5] for the case of a rigid body immersed in a fluid. They showed that, in the case of semi-direct Lie algebra, special care is needed to map between the dual space and the original space of the Lie algebra. If C is such an operator, $C: G^*\to G$ on G^*, then the equations representing some mechanical system on G are written as $x=\mathrm{ad}^*_{C(x)}(x)$, the analogue of Hamilton–Jacobi theory. Another approach to the same problem utilizes the procedure of Hamiltonian reduction on the cotangent bundle of the Lie Group. Such a methodology was adopted by Ratiu [5] while discussing dynamical systems on semi-direct products of Lie algebras. The system of equations constructed is a Hamiltonian system on all orbits of the co-adjoint representation. But if we do not have any condition on C then we cannot say anything about the complete integrability of such a system.

2. Formulation

Let h be a Lie algebra and H the corresponding group: let $\rho: h \to \mathrm{End}(V)$ be a representation of h in the linear space V; and $\alpha: H \to \mathrm{Aut}(V)$ the corresponding representation of the group. Let $O(x)$ denote the orbits of the action of the group H on V, $X \in V$. We now introduce the linear operator, termed 'sectional operator', $Q: h \to V$, where the vector field $\dot{X}_Q = \rho(QX)X$ will then arise on the orbits. A very special class of sectional operators which form a many-parameter set with two basic parameters $a \in V$, $b \in \mathrm{Ker}\ \phi_a$; $\phi_a h = (\rho h)a$ is important for our applications. Let a be a point in the general position in the sense that the orbit through a has maximal dimension. Let $K = \mathrm{Ker}\ \phi_a \subset H$ be the annihilator of a, and let $b \in K$ be any general element of K. Consider the action of ρb on V. Denote $\mathrm{Ker}(\rho b) \subset V$ by M. It is clear that $a \in M$ and $H = K + K'$, where K' is the algebraic complement of K in H. We may now note that V can be written as $V = M \oplus \mathrm{Im}(\rho b)$. If we consider the intersections, $\phi_a K' \cap M = B$ and $\phi_a K' \cap \mathrm{Im}(\rho b) = R'$ then we observe that $\phi_a K'$ has been decomposed as $\phi_a K' = B + R' + P$. The complementary subspace P can be chosen in many ways. We now consider in $\mathrm{Im}(\rho b)$ the space z which is the algebraic complement of R on $\mathrm{Im}(\rho b)$, that is, $\mathrm{Im}(\rho b) = z + R'$. If T is the complement of B in M, then at once we may set

$$V = T + B + R' + Z \qquad R = (\rho b)^{-1} R'$$

where $(\rho b)^{-1}$ is the operator which is inverse to ρb on $\mathrm{Im}(\rho b)$. So $K' = \tilde{B} + \tilde{R} + \tilde{P}$, where $\tilde{B} = \phi_a^{-1} B$, $\tilde{R} = \phi_a^{-1} R$, $\tilde{P} = \phi_a^{-1} P$.

So the sectional operator Q can be defined as

$$Q: V \to H,\ Q: T + B + R + Z \to K + \tilde{B} + \tilde{R} + \tilde{P}.$$

By setting;

$$Q = \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & \phi_a^{-1} & 0 & 0 \\ 0 & 0 & \phi_a^{-1}(\rho b) & 0 \\ 0 & 0 & 0 & D' \end{pmatrix}. \tag{1}$$

Here D is the operator generating the map $T \to K$ and D' gives $Z \to \tilde{P}$. If $V = H^*$, $\rho = \mathrm{ad}^*$; $H \to \mathrm{End}(H^*)$ then

$$\phi_a^{-1} \rho(b) = \phi_a^{-1}\ \mathrm{ad}_a^*. \tag{2}$$

3. Sectional operators and rigid-body dynamics

The methodology of sectional operators can be profitably used to discuss the dynamics of a rigid body. The advantage of such a formulation is that of generalized group-theoretic methods being used to tackle actual physical problems. To this end, let $\mathcal{G}$ be a semi-simple Lie algebra, $B(X, Y)$ be the Cartan-killing form, and f be a smooth function on $\mathscr{G}$. Let us associate with this function a dynamical system on the cotangent bundle $T*G$ of the group by extending f to a left-invariant function F defined on the whole space $T*G$. Let grad f be a field which is dual to the differential df, i.e. for the killing form we obtain $B(\nabla f, \xi)=\xi(f)$. Then the Euler equations can be written as

$$\dot{X}=[X, \nabla f(X)]. \tag{3}$$

Let $\mathscr{G}=SO(n)$ be the Lie algebra of an orthogonal group and let us take the diagonal real matrix

$$I=\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \qquad \lambda_i \neq \lambda_j \qquad i \neq j. \tag{4}$$

We consider on $SO(n)$ the operator $\Psi(X)=IX+XI$. Then the set of equations

$$\Psi\dot{X}=[X, \Psi X] \tag{5}$$

is called the equation of motion of an n-dimensional rigid body. If we represent the algebra $SO(n)$ as the algebra of skew-symmetric real-valued matrices $X=(x_{ij})$ then $\Psi(X)=((\lambda_i+\lambda_j)x_{ij})$. This immediately gives

$$\dot{x}_{ij}=\left(\frac{\lambda_j-\lambda_i}{\lambda_j+\lambda_i}\right)\sum_{q=1}^{n} x_{iq}x_{qj}. \tag{6}$$

Putting $n=3$, we find that these equations coincide with the classical equations of motion of a three-dimensional rigid body.

We can define the analogues of the equations of motion of a rigid body on an arbitrary semi-simple Lie algebra. We can construct a many-parameter set of the operators $\phi: G \to G$ not only for the complex semi-simple Lie algebras but also for their real compact and normal forms. Then all systems described by the equations $\dot{X}=[X, \phi X]$ are completely integrable on the orbits in general position and therefore their integrals define complete commutative sets of functions on both semi-simple Lie algebras and on their real forms.

To make contact with the physics more pronounced, we now turn to the equations of inertial motion of a multi-dimensional rigid body in an ideal fluid using the formalism of Lie theory.

Let $E(n)$ be the Lie group of rigid motions of $\mathbb{R}^n$. $E(n)$ is the semi-direct product of the group $SO(n)$ and the commutative subgroup $\mathscr{J}$ of translations. The matrix representation of the group $E(n)$ has the form [7].

$$\left(\begin{array}{c|c} SO(n) & \begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix} \\ \hline 0 \ \dots \ 0 & 1 \end{array}\right) \qquad (x_1, x_2, \dots, x_n) \in \mathscr{J}. \tag{7}$$

The Lie algebra $\xi(n)$ of the group $E(n)$ is a semi-direct sum $SO(n) \oplus_\varphi \mathbb{R}^n$, $\phi: SO(n) \to \mathrm{End}\, \mathbb{R}^n$ is the differential of the standard representation of $SO(n)$ in $\mathbb{R}^n$, $\mathbb{R}^n$

is regarded as a commutative subalgebra, and $\xi(n)$ has the matrix form

$$\left(\begin{array}{c|c} SO(n) & \begin{matrix} y_1 \\ \vdots \\ y_n \end{matrix} \\ \hline 0 \ \dots \ 0 & 0 \end{array}\right). \tag{8}$$

The commutation rule in the Lie algebra $\xi(n)$ has the form

$$[(x, u), (y, v)] = ([x, y], x(v) - y(u)) \tag{9}$$

where $x, y \in SO(n)$ and $u, v, \in \mathbb{R}^n$.

If we let $B(X, Y)$ be the killing form on $SO(n)$ and $(X, Y)_e$ be the Euclidean scalar product in $\mathbb{R}^n$, then the pairing is given by,

$$((x_1, u_1), (x_2, u_2)) = B(x_1, x_2) + (u_1, u_2)_e. \tag{10}$$

Let us identify the space $\xi(n)^*$ which is dual to $\xi(n)$ with $\xi(n)$. To complete our formalisms we compute the form of the coadjoint action ad* under the isomorphism $\xi(n)^* = \xi(n)$. We use the following definition of the coadjoint action,

$$\{\mathrm{ad}_x^* y, Z\rangle = \langle y, \mathrm{ad}_x Z\rangle \tag{11}$$

where ad denotes the usual adjoint action in the Lie algebra. In our case, let $\chi \in SO(n), u \in \mathbb{R}^n$

$$y \in SO(n)^* = SO(n) \qquad v \in (\mathbb{R}^n)^* = \mathbb{R}^n$$

then using (11) we obtain,

$$\begin{aligned} a((y, v)(x, u))|_{SO(n)} &= [y, x] + \tfrac{1}{2}(vu^t - uv^t) \\ a((y, v)(x, u))|_{\mathbb{R}^n} &= -xv \end{aligned} \tag{12}$$

where we have used the notation $a(x, \xi) = \mathrm{ad}_\xi^*(x)$, $\xi \in \mathscr{G}$, $x \in \mathscr{G}^*$. This incidentally shows why our our approach is different from the usual one via the AKS theorem.

Let $\mathscr{G}$ be an arbitrary Lie algebra. The Euler equations on $\mathscr{G}^*$ are the system of differential equations $\dot{x} = a(x, C(x))$, $C: \mathscr{G}^* \to \mathscr{G}$ being a linear operator, $a(x, \xi)$ being the linear functional. On the orbits of the coadjoint representation ad* of the group G, the Euler equations are the Hamiltonian ones with respect to the canonical symplectic structure.

We are now ready to construct sectional operators for the coadjoint representation of the algebra $\xi(n)$. We have a many-parameter set of sectional operators $Q: \xi(n)^* \to \xi(n)$ for which the Euler equations are completely integrable Hamiltonian systems on the orbits of the coadjoint representation of the group $E(n)$. Let us consider in $\xi(n)^*$ (and hence, in $\xi(n)$) the subspace $[(n+1)/2] - 2$

$$K = \bigoplus_{i=0} \mathbb{R}(e_{2i+1,2i+2}) \oplus \mathbb{R} u_n \subset \xi(n) \tag{13}$$

where e_{ij} is the elementary skew-symmetric matrices, and u_i the standard orthonormal basis in $\mathbb{R}^n$. Denote the corresponding subspace in $\xi(n)^*$ by K^*. Let $K^\perp$ denote the orthogonal complement to the subspace K in $\xi(n)$ or $\xi(n)^*$ with respect to the scalar product defined earlier.

Let $a \in K^*$. Let us consider the mapping $\phi_a : \xi(n) \to \xi(n)^*$, $x \to a(a, x) \in \xi(n)^*$. Then $\phi_a K^{\perp} \subset K^{*\perp}$, where $K^{\perp}$ is the orthogonal complement to K and similarly for $K^{*\perp}$. Note that if a is in the general position, then $K = \text{Ker}\ \phi_a$ and $\phi_a : K^{\perp} \to K^{*\perp}$ is an isomorphism. Hence, the inverse mapping $\phi_a^{-1} : K^{*\perp} \to K^{\perp}$ is defined.

Let us write $\xi(n)$ as the direct sum of the linear spaces $\xi(n) = K^{\perp} \oplus K$. Following the general method, let $a \in K^*$, $b \in K$, a being in the general position. If $Z = x + y \in \xi(n)^*$, $x \in K^{*\perp}$, $y \in K$, then we have the expression for the sectional operator as,

$$\begin{aligned} &Q(a, b, D)z = \phi_a^{-1}\ \text{ad}_b^* x + D(y) \\ &D : K^* \to K \quad \text{being arbitrary.} \end{aligned} \tag{14}$$

We are now in a position to write down the general equations;

$$\dot{x}_Q = \text{ad}^*_{Q(x)}(X) \tag{15}$$

on

$$\mathscr{G}^* = (SO(n) \oplus \mathbb{R}^n) \cong SO(n) \oplus \mathbb{R}^n$$

where $Q(a, b, D)$ is the sectional operator which is a non-compact analogue of the operators $\phi_{a,b,D}$ describing the motion of a rigid body. We have $Q(a, b, D) : \xi(n)^* \to \xi(n)$. Hence $\dot{X}_Q$ is defined on the space $\mathscr{G}^*$. The equations (15) can be written explicitly as:

$$\begin{aligned} &\dot{Y} = [Y, x] + \tfrac{1}{2}(vu' - uv') \\ &\dot{v} = -xv \end{aligned} \tag{16}$$

where we have the explicit dependence

$$(x, u) = Q(a, b, D)(y, v) \tag{17}$$

because the operator $Q(a, b, D)$ has been defined explicitly, we can now say, following reference [5], that the system of differential equations

$$\dot{X} = \text{ad}^*_{Q(a,b,D)X}(X)$$

on the co-algebra $\xi(n)^*$, written explicitly as ( ) is a Hamiltonian algebra on the orbits of the coadjoint representation ad* $(E(n))$. When $n = 3$, the system coincides with the equations of inertial motion of a rigid body in an ideal fluid. Generalizing in the spirit of the above discussion, we can assert that the equations (16) describe the inertial motion of a multi-dimensional rigid body in an ideal fluid for any n.

For the sake of clarity, let us discuss, although briefly, the classical equations of inertial motion of a three-dimensional rigid body in an ideal fluid. Introduce a moving frame of reference. Let u_i be the components of the linear velocity of the origin in the moving frame. Alternatively, in a fixed frame of reference, u_i may be thought of as components of the fluid velocity. Also let ω_i be the components of angular velocity of rotation of the rigid body in the fluid. The kinetic energy of the body and fluid system has the form

$$T = \tfrac{1}{2}(A_{ij}\omega_i\omega_j + B_{ij}u_iu_j) + C_{ij}\omega_iu_j \tag{18}$$

A_{ij}, B_{ij}, C_{ij} being constants depending on the body form and the density of the body and the fluid. Let

$$M=(y_1, y_2, y_3) \qquad y_i=\frac{\partial T}{\partial \omega_i} \tag{19}$$

$$L=(x_1, x_2, x_3) \qquad x_i=\frac{\partial T}{\partial u_i}. \tag{20}$$

Then the inertial motion of a rigid body in an ideal fluid can be completely described by the equations

$$\frac{\mathrm{d}M}{\mathrm{d}t}=M\times\omega+L\times U \tag{21}$$

$$\frac{\mathrm{d}L}{\mathrm{d}t}=L\times\omega \tag{22}$$

where $U=(u_1, u_2, u_3)$ and $\omega=(\omega_1, \omega_2, \omega_3)$. For more discussion on this point and related questions, we refer to reference [5].

Let us now outline the motivation for the present work. Although the sectional operator technique has been used to discuss the case of a three-dimensional rigid body in an ideal fluid, no higher-dimensional case has as yet been discussed. We therefore present a systematic and extended investigation for a higher-dimensional case ($n=6$) and construct new integrable dynamical systems. Manakov [8] has already dealt with the pure $SO(6)$ case. Our approach is, in a sense, a generalization of his work. Moreover, we postulate that our equations contain the system of two interacting rigid bodies immersed in an ideal fluid as a special case. In this way, we have also extended the work done in [5].

Lastly, let us mention that, although the algebra $SO(n)\times V_6$ is a simple Inonu–Wigner contraction of $SO(7)$ with respect to the coordinate 7, we have not endeavoured to tackle our problem that way, because the whole aim of the present work is to construct new integrable dynamical systems related to some physical problem and not merely to present some mathematical artifice. Integrability is guaranteed in the sectional operator approach. Hence we feel that the approach we have adopted is an elegant and useful one apart from being a natural complement to the usual methods to obtain integrable systems like the AKNS and AKS formalisms.

4. Construction of Q in case of non-compact algebra $SO(6)+V_6$

$SO(6)$ is the real Lie algebra of all 6×6 real antisymmetric matrices [9]. Let us introduce 6×6 real antisymmetric matrices M_{pq} defined by

$$(M_{pq})_{jk}=\delta_{pj}\delta_{qk}-\delta_{pk}\delta_{qj} \qquad j, k, p, q=1, 2, \ldots, 6$$

i.e.

$$M_{ij}=e_{ij}-e_{ji} \tag{23}$$

where e_{ij} is the matrix with the unit element in the intersection of the ith row and jth column and zero elsewhere.

The Lie algebra $SO(6)$ is defined by the commutation relation

$$[M_{pq}, M_{rs}] = \delta_{Qr} M_{rs} - \delta_{qs} M_{pr} - \delta_{pr} M_{qs} + \delta_{ps} M_{qr}. \tag{24}$$

Let e_l $(L=1, \ldots, 6)$ denote the basis vectors in the vector space V_6. With the representation of $SO(6)$ introduced above we can say that V_6 is the topological vector space on which the representation of $SO(6)$ is realized; of course, V_6 also serves as an invariant subalgebra of $SO(6) \times V_6$. The role of V_6 becomes quite clear in the context of our discussion in section 3.

We start with the identification of the subspaces K and K^l.

$$\begin{aligned} K = K^* &= (a_1 M_{12} + a_2 M_{34}) \oplus a_3 e_6 \\ &= A_1 \oplus A_2 \qquad \text{(say)} \end{aligned} \tag{25}$$

$$\begin{aligned} K_1 = K^{*l} = (& f_3 M_{13} + f_4 M_{14} + f_5 M_{15} + f_6 M_{16} + f_7 M_{23} + f_8 M_{24} + f_9 M_{25} \\ &+ f_{10} M_{26} + f_{11} M_{35} + f_{12} M_{36} + f_{13} M_{45} + f_{14} M_{46} + f_{15} M_{56}) \\ &\oplus (u_1 e_1 + u_2 e_2 + u_3 e_3 + u_4 e_4 + u_5 e_5) \\ = F_1 \oplus F_2 & \qquad \text{(say)}. \end{aligned} \tag{26}$$

If we denote by the letter 'a' the element of general form $\in K$ and f denotes the corresponding element $\in K^1$, then we get at once

$$\mathrm{ad}_a^* f = ([F_1, A_1] + \tfrac{1}{2}(F_2 A_2^t - A_2 F_2^t), -A_1 F_2) \tag{27}$$

$$= \begin{pmatrix} \lambda & \mu \\ \nu & \epsilon \end{pmatrix} \oplus \begin{pmatrix} -a_1 u_2 \\ a_1 u_1 \\ -a_2 u_2 \\ a_2 u_3 \\ 0 \\ 0 \end{pmatrix} \tag{28}$$

where $(\lambda, \mu, \nu, \epsilon)$ each is given 3×3 matrices written below:

$$\lambda = \begin{pmatrix} 0 & 0 & -(f_4 a_2 + f_7 a_1) \\ 0 & 0 & -f_8 a_2 + a_1 f_3 \\ f_7 a_1 + f_4 a_2 & -f_3 a_1 + f_8 a_2 & 0 \end{pmatrix} \tag{29a}$$

$$\nu = \begin{pmatrix} f_8 a_1 - a_2 f_3 & -f_4 a_1 - a_2 f_7 & 0 \\ f_9 a_1 & -f_5 a_1 & f_{13} a_2 \\ f_{10} a_1 - \frac{1}{2} u_1 a_3 & -f_6 a_1 - \frac{1}{2} u_2 a_3 & f_{14} a_2 - \frac{1}{2} u_3 a_3 \end{pmatrix} \tag{29b}$$

$$\mu = \begin{pmatrix} f_3 a_2 - f_8 a_1 & -f_9 a_1 & -f_{10} a_1 + \frac{1}{2} u_1 a_3 \\ f_7 a_2 + a_1 f_4 & a_1 f_5 & a_1 f_6 + \frac{1}{2} u_2 a_3 \\ 0 & -f_{13} a_2 & -f_{14} a_2 + \frac{1}{2} u_3 a_3 \end{pmatrix} \tag{29c}$$

and

$$\epsilon=\begin{pmatrix} 0 & f_{11}a_2 & f_{12}a_2+\frac{1}{2}u_4a_3 \\ -f_{11}a_2 & 0 & \frac{1}{2}u_5a_3 \\ -f_{12}a_2-\frac{1}{2}u_4a_3 & -\frac{1}{2}u_5a_3 & 0 \end{pmatrix}. \tag{29d}$$

Let us now consider two general elements $b\in K$ and $x\in K^1$ with the structures given in equations (25) and (26). For example, b is written as

$$b=(b_1M_{12}+b_2M_{34})\oplus b_3e_6=B_1\oplus B_2. \tag{30}$$

Similarly, two parts of x are denoted as X_1 and X_2. Whence we get

$$\phi_b(x)=\mathrm{ad}_x^*b=([B_1,X_1]+\tfrac{1}{2}(B_2X_2'-X_2B_2'),-X_1B_2) \tag{31}$$

which can immediately be evaluated as in equation (27). We at once obtain

$$\phi_b^{-1}\,\mathrm{ad}_a^*f=\begin{pmatrix} P & Q \\ R & S \end{pmatrix}\oplus\begin{pmatrix} p \\ q \end{pmatrix} \tag{32}$$

where (P,Q,R,S) each is a block of 3×3 matrices and p,q are three component vectors, all given below:

$$P=\begin{pmatrix} 0 & 0 & \lambda_1f_3-\lambda_2f_8 \\ 0 & 0 & \lambda_2f_4+\lambda_1f_7 \\ -(\lambda_1f_3-\lambda_2f_8) & -(\lambda_2f_4+\lambda_1f_7) & 0 \end{pmatrix} \tag{33a}$$

$$Q=\begin{pmatrix} \lambda_1f_4+\lambda_2f_7 & -f_5\dfrac{a_1}{b_1} & \dfrac{a_1}{b_3}u_2 \\ \lambda_1f_8-\lambda_2f_3 & -f_9\dfrac{a_1}{b_1} & -\dfrac{a_1}{b_3}u_1 \\ 0 & -f_{11}\dfrac{a_2}{b_2} & \dfrac{a_2}{b_3}u_4 \end{pmatrix} \tag{33b}$$

$$R=\begin{pmatrix} -(\lambda_1f_4+\lambda_2f_7) & -(\lambda_1f_8-\lambda_2f_3) & 0 \\ f_5\dfrac{a_1}{b_1} & f_9\dfrac{a_1}{b_1} & f_{11}\dfrac{a_2}{b_2} \\ -\dfrac{a_1}{b_3}u_2 & \dfrac{a_1}{b_3}u_1 & -\dfrac{a_2}{b_3}u_4 \end{pmatrix} \tag{33c}$$

$$S=\begin{pmatrix} 0 & -f_{13}\dfrac{a_2}{b_2} & -\dfrac{a_2}{b_3}u_3 \\ f_{13}\dfrac{a_2}{b_2} & 0 & 0 \\ \dfrac{a_2}{b_3}u_3 & 0 & 0 \end{pmatrix} \tag{33d}$$

$$p=\begin{pmatrix} 2\dfrac{a_1}{b_3}\left(f_{10}-\dfrac{b_1}{b_3}u_1\right)-u_1\dfrac{a_3}{b_3} \\ -2\dfrac{a_1}{b_3}\left(f_6+\dfrac{b_1}{b_3}u_2\right)-u_2\dfrac{a_3}{b_3} \\ 2\dfrac{a_2}{b_3}\left(f_{14}-\dfrac{b_2}{b_3}u_3\right)-u_3\dfrac{a_3}{b_3} \end{pmatrix} \tag{33e}$$

$$q=\begin{pmatrix} -2\dfrac{a_2}{b_3}\left(f_{12}+\dfrac{b_2}{b_3}u_4\right)-u_4\dfrac{a_3}{b_3} \\ -\dfrac{a_3}{b_3}u_5 \\ 0 \end{pmatrix} \tag{33f}$$

where

$$\lambda_1=\frac{a_2b_2-a_1b_1}{b_1^2-b_2^2} \qquad \text{and} \qquad \lambda_2=\frac{a_1b_2-a_2b_1}{b_1^2-b_2^2}. \tag{34}$$

So finally the form of the sectional operator $Q(a, b, D)$ is given by

$$Q(a, b, D)[F_1\oplus F_2]=Q_1\oplus Q_2 \tag{35}$$

where

$$Q_2=(\alpha_3 f_1+\beta_3 f_2+\gamma_3 u_6)e_6$$

and

$$Q_1=\begin{pmatrix} Q_1^{11} & Q_1^{12} \\ Q_1^{21} & Q_1^{22} \end{pmatrix} \tag{36}$$

where each Q_1^{ij} is a 3×3 matrix, as given below,

$$Q_1^{11}=\begin{pmatrix} 0 & \alpha_1 f_1+\beta_1 f_2+\gamma_1 u_6 & \lambda_1 f_3-\lambda_2 f_8 \\ -(\alpha_1 f_1+\beta_1 f_2+\gamma_1 u_6) & 0 & \lambda_2 f_4+\lambda_1 f_7 \\ -(\lambda_1 f_3-\lambda_2 f_8) & -(\lambda_2 f_4+\lambda_1 f_7) & 0 \end{pmatrix} \tag{37a}$$

$$Q_1^{12}=\begin{pmatrix} \lambda_1 f_4+\lambda_2 f_7 & -f_5\dfrac{a_1}{b_1} & \dfrac{a_1}{b_3}u_2 \\ \lambda_1 f_8-\lambda_2 f_3 & -f_9\dfrac{a_1}{b_1} & -\dfrac{a_1}{b_3}u_1 \\ \alpha_2 f_1+\beta_2 f_2+\gamma_2 u_6 & -f_{11}\dfrac{a_2}{b_2} & \dfrac{a_2}{b_3}u_4 \end{pmatrix} \tag{37b}$$

$$Q_1^{21}=\begin{pmatrix} -(\lambda_1 f_4+\lambda_2 f_7) & -(\lambda_1 f_8-\lambda_2 f_3) & -(\alpha_2 f_1+\beta_2 f_2+\gamma_2 u_6) \\ f_5\dfrac{a_1}{b_1} & f_9\dfrac{a_1}{b_1} & f_{11}\dfrac{a_2}{b_2} \\ -\dfrac{a_1}{b_3}u_2 & \dfrac{a_1}{b_3}u_1 & -\dfrac{a_2}{b_3}u_4 \end{pmatrix} \tag{37c}$$

$$Q_l^{22}=\begin{pmatrix} 0 & -f_{13}\dfrac{a_2}{b_2} & -\dfrac{a_2}{b_3}u_3 \\ f_{13}\dfrac{a_2}{b_2} & 0 & 0 \\ \dfrac{a_2}{b_2}u_3 & 0 & 0 \end{pmatrix}. \tag{37d}$$

Thus all equations above are crucial in the construction and presentation of the sectional operator $Q(a, b, D)$.

5. Dynamical system

The integrable equation of motion can therefore be written as

$$\begin{aligned}\dot{f}&=\mathrm{ad}^*_{A(a,b,D)}(F)\\ &=([F_1, Q_1]+\tfrac{1}{2}(F_2Q_2^t-Q_2F_2^t), -Q_1F_2).\end{aligned} \tag{38}$$

Evaluating the right-hand side of this equation explicitly with the expression for Q given above, we obtain the equations for the dynamical system constructed on $SO(6)\times V_6$ via the sectional operator approach.

$$\begin{aligned}\dot{u}_1=&-u_2(\alpha_1f_1+\beta_1f_2+\gamma_1u_6)-u_3(\lambda_1f_3-\lambda_2f_8)-u_4(\lambda_1f_4+\lambda_2f_7)\\ &+\frac{a_1}{b_1}f_5u_5-\frac{a_1}{b_3}u_2u_6\end{aligned} \tag{39a}$$

$$\begin{aligned}\dot{u}_2=&+u_1(\alpha_1f_1+\beta_1f_2+\gamma_1u_6)-u_3(\lambda_2f_4+\lambda_1f_7)-u_4(\lambda_1f_8-\lambda_2f_3)\\ &+\frac{a_1}{b_1}f_9u_5+\frac{a_1}{b_3}u_1u_6\end{aligned} \tag{39b}$$

$$\begin{aligned}\dot{u}_3=&u_1(\lambda_1f_3-\lambda_2f_8)+u_2(\lambda_2f_4+\lambda_1f_7)-u_4(\alpha_2f_1+\beta_2f_2+\gamma_2u_6)\\ &+\frac{a_2}{b_2}f_{11}u_5-\frac{a_2}{b_3}u_4u_6\end{aligned} \tag{39c}$$

$$\begin{aligned}\dot{u}_4=&u_1(\lambda_1f_4+\lambda_2f_7)+u_2(\lambda_1f_8-\lambda_2f_3)+u_3(\alpha_2f_1+\beta_2f_2+\gamma_2u_6)\\ &+\frac{a_2}{b_2}f_{13}u_5+\frac{a_2}{b_3}u_3u_6\end{aligned} \tag{39d}$$

$$\dot{u}_5=\frac{a_1}{b_1}u_1f_5+\frac{a_1}{b_1}u_2f_9+\frac{a_2}{b_2}f_{11}u_3+\frac{a_2}{b_2}f_{13}u_4 \tag{39e}$$

$$\dot{u}_6=0 \tag{39f}$$

$$\dot{f}_1=0 \qquad \dot{f}_2=0 \tag{39g}$$

$$\dot{f}_3=f_1(\lambda_2 f_4+\lambda_1 f_7)-f_4(\alpha_2 f_1+\beta_2 f_2+\gamma_2 u_6)-f_7(\alpha_1 f_1+\beta_1 f_2+\gamma_1 u_6)$$

$$+f_2(\lambda_1 f_4+\lambda_2 f_7)+\left(\frac{a_2}{b_2}-\frac{a_1}{b_1}\right)f_5 f_{11}+\frac{a_2}{b_3}(u_1 f_{14}-u_4 f_6)$$

$$+\frac{a_1}{b_3}(u_2 f_{12}-u_3 f_{10})+\frac{a_1 b_1-a_2 b_2}{b_3^2}u_1 u_3 \qquad (39h)$$

$$\dot{f}_4=f_1(\lambda_1 f_8-\lambda_2 f_3)-f_2(\lambda_1 f_3-\lambda_2 f_8)+f_3(\alpha_2 f_1+\beta_2 f_2+\gamma_2 u_6)-f_8(\alpha_1 f_1+\beta_1 f_2+\gamma_1 u_6)$$

$$+\frac{a_2}{b_2}(f_5 f_{13}+u_3 f_6)-\frac{a_1}{b_1}f_5 f_{13}+\frac{a_1}{b_3}(u_2 f_{14}-u_4 f_{10})$$

$$-\frac{a_2}{b_3}u_1 f_{12}+\frac{a_1 b_1-a_2 b_2}{b_3^2}u_1 u_4 \qquad (39i)$$

$$\dot{f}_5=-f_{11}(\lambda_1 f_3-\lambda_2 f_8)-f_{13}(\lambda_1 f_4+\lambda_2 f_7)-f_9(\alpha_1 f_1+\beta_1 f_2+\gamma_1 u_6)$$

$$-f_1 f_9\frac{a_1}{b_1}-(f_3 f_{11}+f_4 f_{13})\frac{a_2}{b_2}+\frac{a_1}{b_3}(u_2 f_{15}-u_5 f_{10})+\frac{a_1 b_1}{b_3^2}u_1 u_5 \qquad (39j)$$

$$\dot{f}_6=-f_{12}(\lambda_1 f_3-\lambda_2 f_8)-f_{14}(\lambda_1 f_4+\lambda_2 f_7)-f_{10}(\alpha_1 f_1+\beta_1 f_2+\gamma_1 u_6)$$

$$+\tfrac{1}{2}u_1(\alpha_3 f_1+\beta_2 f_2+\gamma_3 u_6)+f_5 f_{15}\frac{a_1}{b_1}+\frac{a_2}{b_3}(f_3 u_4-f_4 u_3)-\frac{a_1}{b_3}(f_1 u_1+u_6 f_{10})$$

$$+\frac{1}{2}\frac{a_3}{b_3}u_1 u_6+\frac{a_1 b_1}{b_3^2}u_1 u_6 \qquad (39k)$$

$$\dot{f}_7=-f_1(\lambda_1 f_3-\lambda_2 f_8)+f_2(\lambda_1 f_8-\lambda_2 f_3)-f_8(\alpha_2 f_1+\beta_2 f_2+\gamma_2 u_6)$$

$$+f_3(\alpha_1 f_1+\beta_1 f_2+\gamma_1 u_6)+\frac{a_2}{b_2}f_9 f_{11}-\frac{a_1}{b_1}f_9 f_{11}+\frac{a_2}{b_3}(u_2 f_{14}-u_4 f_{10})$$

$$+\frac{a_1}{b_3}(u_3 f_6-u_1 f_{12})+\frac{a_1 b_1-a_2 b_2}{b_3^2}u_2 u_3 \qquad (39l)$$

$$\dot{f}_8=-f_1(\lambda_1 f_4+\lambda_2 f_7)-f_2(\lambda_2 f_4+\lambda_1 f_7)+f_7(\alpha_2 f_1+\beta_2 f_2+\gamma_2 u_6)$$

$$+f_4(\alpha_1 f_1+\beta_1 f_2+\gamma_1 u_6)+\left(\frac{a_1}{b_2}-\frac{a_1}{b_1}\right)f_9 f_{13}$$

$$+\frac{a_2}{b_3}(u_3 f_{10}-u_2 f_{12})+\frac{a_1}{b_3}(u_4 f_6-u_1 f_{14})+\frac{a_1 b_1-a_2 b_2}{b_3^2}u_2 u_4 \qquad (39m)$$

$$\dot{f}_9 = -f_{11}(\lambda_1 f_7 + \lambda_2 f_4) - f_{13}(\lambda_1 f_8 - \lambda_2 f_3) + f_5(\alpha_1 f_1 + \beta_1 f_2 + \gamma_1 u_6) + \frac{a_1}{b_1} f_1 f_5$$
$$- \frac{a_2}{b_2}(f_7 f_{11} + f_8 f_{13}) + \frac{a_1}{b_3}(u_5 f_6 - u_1 f_{15}) + \frac{a_1 b_1}{b_3^2} u_2 u_5 \tag{39n}$$

$$\dot{f}_{10} = -f_{12}(\lambda_1 f_7 + \lambda_2 f_4) - f_{14}(\lambda_1 f_8 - \lambda_2 f_3) + f_6(\alpha_1 f_1 + \beta_1 f_2 + \gamma_1 u_6)$$
$$+ \tfrac{1}{2} u_2(\alpha_3 f_1 + \beta_2 f_2 + \gamma_3 u_6) + f_{15} f_9 \frac{a_1}{b_1} + \frac{a_1}{b_3}(u_6 f_6 - u_2 f_1)$$
$$+ \frac{a_2}{b_3}(u_4 f_7 - u_3 f_3 + \tfrac{1}{2} u_2 u_6) \frac{a_3}{b_3} + \frac{a_1 b_1}{b_3^2} u_2 u_6 \tag{39o}$$

$$\dot{f}_{11} = f_5(\lambda_1 f_3 - \lambda_2 f_8) + f_9(\lambda_2 f_4 + \lambda_1 f_7) - f_{13}(\alpha_2 f_1 + \beta_2 f_2 + \gamma_2 u_6)$$
$$+ \frac{a_1}{b_1}(f_3 f_5 + f_7 f_9) - \frac{a_1}{b_1} f_2 f_{13} + \frac{a_2}{b_3}(u_4 f_{15} - u_5 f_{14}) + \frac{a_2 b_2}{b_3^2} u_3 u_5 \tag{39p}$$

$$\dot{f}_{12} = f_6(\lambda_1 f_3 - \lambda_2 f_8) + f_{10}(\lambda_2 f_4 + \lambda_1 f_7) - f_{14}(\alpha_2 f_1 + \beta_2 f_2 + \gamma_2 u_6)$$
$$+ \tfrac{1}{2} u_3(\alpha_3 f_{11} + \beta_3 f_2 + \gamma_3 u_6) + \frac{a_1}{b_3}(u_1 f_7 - u_2 f_3) - \frac{a_2}{b_3}(u_3 f_2 + u_6 f_{14})$$
$$+ \frac{a_2}{b_2} f_{11} f_{15} + \frac{a_3}{2b_3} u_3 u_6 + \frac{a_2 b_2}{b_3^2} u_3 u_6 \tag{39q}$$

$$\dot{f}_{13} = f_5(\lambda_1 f_4 + \lambda_2 f_7) + f_9(\lambda_1 f_8 - \lambda_2 f_3) + f_{11}(\alpha_2 f_1 + \beta_2 f_2 + \gamma_2 u_6)$$
$$+ \frac{a_1}{b_1}(f_4 f_5 + f_8 f_9) + \frac{a_2}{b_2} f_2 f_{11} + \frac{a_2}{b_3}(u_5 f_{12} - u_3 f_{15}) + \frac{a_2 b_2}{b_3^2} u_4 u_5 \tag{39r}$$

$$\dot{f}_{14} = f_6(\lambda_1 f_4 + \lambda_2 f_7) + f_{10}(\lambda_1 f_8 - \lambda_2 f_3) + f_{12}(\alpha_2 f_1 + \beta_2 f_2 \gamma_2 u_6)$$
$$+ \frac{u_4}{2}(\alpha_3 f_1 + \beta_3 f_2 + \gamma_3 u_6) + f_{13} f_{15} \frac{a_2}{b_2} + \frac{a_1}{b_3}(u_1 f_8 - u_2 f_4)$$
$$+ \frac{a_2}{b_3}(u_6 f_{12} - u_4 f_2) + \tfrac{1}{2} u_4 u_6 \frac{a_3}{b_3^2} + \frac{a_2 b_2}{b_3^2} u_6 u_4 \tag{39s}$$

$$\dot{f}_{15} = \frac{u_5}{2}(\alpha_3 f_1 + \beta_3 f_2 + \gamma_3 u_6) + \frac{a_1}{b_3}(u_1 f_9 - u_2 f_5) + \frac{a_2}{b_3}(u_3 f_{13} - u_4 f_{11})$$
$$- \frac{a_1}{b_1}(f_5 f_6 + f_9 f_{10}) - \frac{a_2}{b_2}(f_{11} f_{12} + f_{13} f_{14}) + \frac{1}{2} \frac{a_3}{b_3} u_5 u_6. \tag{39t}$$

So we have 21 coupled nonlinear equations which reduce to those of Manakov for $n=6$ and if all $u_i=0$.

5. Discussions

In our above analysis we have shown how extended dynamical systems can be constructed on non-compact Lie algebras, keeping intact its integrability property. This approach depends on the construction of the sectional operator introduced initially by Fomenko and Trofimov. Our equations give a generalization of those of Manakov for pure $SO(6)$ and also those of a rigid body immersed in a fluid which deals with the case $SO(3)\times V_3$. The Hamiltonian of the present system is given by the invariant $\langle f, Q(a,b,D)(f)\rangle$ and from its explicit structure it is not difficult to read off the Poisson structure.

References

[1] Adler M and Van Moerbeke P 1980 *Adv. Math.* **38** 267 Reyman, Semenov A G and Tian Shansky M A 1981 *Invent. Math.* **63** 423
[2] Adams M R, Harnad J and Previato E 1988 *Commun. Math. Phys.* **117** 451
Flascka H, Newell A C and Ratiu T 1995 *Physica* **D**
[3] Adler M 1979 *Invent. Math.* **50** 219
Kostant B 1979 *Adv. Math.* **34** 195
Symes W 1980 *Invent Math.* **159** 13
[4] Trofimov V V and Fomenko A T 1983 *Funct. Anal. Appl.* **17** 23
[5] Fomenko A T and Trofimov V V 1987 *Integrable Systems, Lie Algebra and Symmetric Spaces* (London, Paris: Gordon and Breach)
[6] Ratiu T 1980 *American J. Math.* **104** 409
Ratiu T 1981 *Trans. American Math. Soc.* **264** 321
[7] Guillemin V and Sternberg S 1984 *Symplectic Techniques in Physics* (Cambridge: Cambridge University Press)
[8] Manakov S V 1976 *Funct. Anal. Appl.* **10** 328
[9] Cornwell J F 1984 *Group Theory in Physics* vol II (London: Academic)